Long paths and cycles in subgraphs of the cube

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Abstract

Let Q_n denote the graph of the n-dimensional cube with vertex set $\{0,1\}^n$ in which two vertices are adjacent if they differ in exactly one coordinate. Suppose G is a subgraph of Q_n with average degree at least d. How long a path can we guarantee to find in G?

Our aim in this paper is to show that G must contain an exponentially long path. In fact, we show that if G has minimum degree at least d then G must contain a path of length $2^d - 1$. Note that this bound is tight, as shown by a d-dimensional subcube of Q_n. We also obtain the slightly stronger result that G must contain a cycle of length at least 2^d.

1 Introduction

Given a graph G of average degree at least d, a classical result of Dirac [4] guarantees a path of length d in G. Moreover, this bound is best possible as can be seen from K_{d+1}.

Inside the cube Q_n can we improve this bound? That is, given a subgraph G of Q_n with average degree at least d, what is the length of the longest path in G? The edge isoperimetric inequality for the cube ([1], [5], [6], [7], see [2] for background) says that any subgraph of average degree at least d must have size at least 2^d. In light of this, the above linear bound seems very weak. A natural subgraph of Q_n with average degree at least d is the d-dimensional cube Q_d, the analogue of the complete graph in Q_n, which contains a path of length $2^d - 1$. Must the size of the longest path in G also be exponential?

The main result of this paper answers this question in the affirmative.

Theorem 1.1. *Every subgraph G of Q_n with minimum degree d contains a path of length $2^d - 1$.*

Note that this is best possible as shown by a d-dimensional subcube of Q_n. In fact, the proof of Theorem 1.1 shows that we can always find a longer path in G unless it is isomorphic to Q_d. Using the well known fact that every graph with average degree at least d contains a subgraph with minimum degree at least $\frac{d}{2}$ we obtain the following corollary to Theorem 1.1.

Corollary 1.2. *Every subgraph G of Q_n with average degree at least d contains a path of length at least $2^{\frac{d}{2}} - 1$.*

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We do not know a tight bound for average degree d. We also obtain the corresponding result for the length of the longest cycle in subgraphs of Q_n with large minimum degree.

Theorem 1.3. *Every subgraph G of Q_n with minimum degree d contains a cycle of length at least 2^d.*

In Section 2 we give an overview of the proofs of Theorems 1.1 and 1.3. The theorems themselves are then proved in Sections 3-7.

In Section 8 we show that the lower bound from Theorems 1.1 and 1.3 also extends to subgraphs of the grid graph $\mathbb{Z}^n$ and the discrete torus C_k^n, for all $k \geq 4$. We also give a generalization of Theorems 1.1 and 1.3 to general 'product-type' graphs in the following form:

Theorem 1.4. *Let $k \in \mathbb{N}$. Suppose G is a graph with minimum degree at least d and that G has the following property:*

> *Given any two vertices $x, y \in G$, there is a partition of $V(G)$ into two sets X and Y with $x \in X$ and $y \in Y$ such that $d_{G[X]}(v) \geq d(v) - k$ for all $v \in X$ and $d_{G[Y]}(v) \geq d(v) - k$ for all $v \in Y$.*

Then G contains a path of length at least $2^{\frac{d}{k+2}}$.

In Section 8 we also give some consequences of this theorem and make some conjectures.

2 Overview

As in the statement of Theorem 1.1, let G be a subgraph of Q_n with $\delta(G) \geq d$. We will view the vertices of Q_n as elements of the power set of $[n]$, $\mathcal{P}[n]$.

A plausible approach to proving Theorem 1.1 is to split G along some direction i to obtain two induced subgraphs G_1 and G_2 consisting of those vertices of G respectively containing and not containing i, for some $i \in [n]$. Provided such a direction is chosen to ensure that $G_1, G_2 \neq \emptyset$, we have $\delta(G_1), \delta(G_2) \geq d - 1$ and by induction on Theorem 1.1 we have a path of length $2^{d-1} - 1$ in each subgraph. If we could join these two paths into one we would clearly be done. However, as Theorem 1.1 provides no information on where these paths start or end, we can not expect to be able to do this.

This suggests that we strengthen Theorem 1.1 to guarantee an exponentially long path between *any* two vertices x and y of G. In general this is not possible – for example, consider the graph G' obtained by removing all but one edge xy of direction $d + 1$ from the $(d + 1)$-dimensional cube Q_{d+1}.

However this graph is not 2-connected. The following theorem says that this is the only obstruction to such a strengthening.

Theorem 2.1. *Let G be a 2-connected subgraph of Q_n and a and b be distinct vertices of G. Suppose that $d_G(z) \geq d$ for all $z \in G - \{a, b\}$. Then G contains an $a - b$ path of length at least $2^d - 2$. Furthermore, unless G is isomorphic to Q_d with a and b at even Hamming distance, G contains an $a - b$ path of length at least $2^d - 1$.*

Note that we do not assume that a or b have degree at least d in Theorem 2.1. This slight weakening of the minimum degree condition will allow us to use induction on various subgraphs of G which would otherwise not be available.

Before continuing with the overview we make a small diversion to introduce some definitions: these are standard (e.g. see [3]).

A subgraph B of a graph G is a *block* of G if B is either a bridge of G or forms a maximal 2-connected subgraph of G. By maximality, $|B_1 \cap B_2| \leq 1$ for any two blocks B_1 and B_2 of G and $G - E(B)$ contains no $x - y$ path between distinct vertices x, y in a block B. Therefore if any two blocks intersect, their common vertex must be a cutvertex and conversely every cutvertex lies in at least two blocks. Since every cycle is 2-connected and an edge is a bridge iff it does not lie in any cycle, every graph G decomposes uniquely into its blocks $B_1, \dots, B_p$ in the sense that:

$$E(G) = \bigcup_{i=1}^{p} E(B_i) \text{ and } E(B_i) \cap E(B_j) = \emptyset \text{ if } i \neq j.$$

Suppose now that G is connected. Let $\mathcal{B}(G)$, the *block-cutvertex graph* of G, be the bipartite graph with bipartition $(\mathcal{B}, \mathcal{C})$ where $\mathcal{B}$ is the set of blocks of G, $\mathcal{C}$ is the set of cutvertices of G with Bc an edge if $c \in B$. For a connected graph G, $\mathcal{B}(G)$ is a tree.

The leaves of this tree are all elements of $\mathcal{B}$ and are called *endblocks*. Given an endblock E we will denote its unique cutvertex by $\operatorname{cutv}(E)$. Note that a graph G has only one endblock iff it is 2-connected.

We now return to the overview of the proof of Theorem 2.1.

Lemma 2.2. *Let G be a connected subgraph of Q_n with a and b distinct vertices of G. Then there exists a partition of G into two connected subgraphs G_a and G_b such that $a \in G_a$, $b \in G_b$ and for all $v \in G_c$, $d_{G_c}(v) \geq d_G(v) - 1$, where $c \in \{a, b\}$.*

Proof. Picking $i \in [n]$ such that a and b differ in coordinate i and forming G_1 and G_2 as before, we have $a \in G_1$ and $b \in G_2$. Let C_b be the connected component of G_2 containing b. Taking G_a to be the connected component of $G - C_b$ containing a and $G_b = G - G_a$ we are done. □

We will refer to i in the above proof as the splitting direction for G_a and G_b.

A central observation in the proof of Theorem 2.1 is that, provided $d \geq 3$, given any endblock E of G_a with $a \notin E$, by induction on Theorem 2.1, E contains a path of length at least $2^{d-1} - 2$ from $\operatorname{cutv}(E)$ to any $y \in E - \operatorname{cutv}(E)$ – $d \geq 3$ here guarantees E is 2-connected and not a bridge. Since G is 2-connected there must exist $y \in E - \operatorname{cutv}(E)$ with a neighbour in G_b. Thus these endblocks guarantee 'endblock paths' of length at least $2^{d-1} - 1$ from a point in G_a to one in G_b. If we could find a path from a to b containing at least two such endblock paths we would almost have our path (it may still be short two or three vertices to give the $2^d - 2$ or $2^d - 1$ bound).

For ease of exposition we will prove the following weakening of Theorem 2.1 first. It will allow the reader to focus on the main ideas in the proof of Theorem 2.1 without some distracting details necessary to ensure that an $a - b$ path formed from endblock paths is not slightly too short.

Theorem 2.3. *Let G be a 2-connected subgraph of Q_n and $a, b \in V(G)$. Suppose that $d_G(z) \geq d$ for all $z \in V(G) - \{a, b\}$. Then G contains an $a - b$ path of length at least 2^{d-1}.*

Another technicality that arises in the proof of Theorem 2.1 and 2.3 is the possibility that the only choice of a splitting direction i for G_a and G_b in Lemma 2.2 above, leaves a with just one neighbour in G_a or b with just one neighbour in G_b. While all cases can be dealt with simultaneously, we felt for clarity's sake it was easier to first restrict attention to the case where a splitting direction i exists for G_a and G_b in which $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$.

Theorem 2.3 is proved in Sections 3-6. Sections 3-5 will focus on the case where we can find a partition direction i, such that $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$. Section 3 will describe the block-cutvertex decomposition structure of G_a and G_b in the absence of an $a - b$ path of length 2^{d-1} formed by joining at least two endblock paths together, and Section 4 describes how the endblocks of G_a interact with those of G_b. In Section 5 we show that if G does not contain a path from a to b containing at least two endblock paths then the conditions of Theorem 2.3 hold for a smaller subgraph of G. This allows for an inductive step and completes the proof of Theorem 2.3 in this case.

Section 6 will allow us, using a small modification of the argument from Sections 3-5, to extend from the case $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$ to the general case, proving Theorem 2.3.

Finally in Section 7 we show how to adjust the approach in Sections 3-6 to obtain the optimal bound of Theorem 2.1.

To close this section we show that Theorem 2.1 implies Theorem 1.3.

Proof of Theorem 1.3: Take an endblock E in the block-cutvertex decomposition of G. Clearly E is 2-connected and all vertices in $E - \text{cutv}(E)$ have at least d neighbours in E. Pick a neighbour v of $\text{cutv}(E)$ in E. Then by Theorem 2.1 G contains a $\text{cutv}(E) - v$ path P of length at least $2^d - 1$. Combining P with the edge $\text{cutv}(E)v$ we obtain the desired cycle. □

3 Endblocks in G_a and G_b

To begin we introduce some useful definitions.

Definition 3.1. Let E be an endblock in the block-cutvertex decomposition of G_a (G_b). The *interior* of E is the set $\text{int}(E) = E - \text{cutv}(E)$. A vertex $x \in \text{int}(E)$ is said to be an *exit vertex* of E if x has a neighbour in G_b (G_a). If this neighbour exists, it is unique and is denoted by $p(x)$, x's *partner*.

Definition 3.2. $\text{Body}(a)$ is the intersection of all blocks of G_a containing a. Let $\text{Core}(a)$ consist of those vertices in $\text{Body}(a)$ that are not cutvertices of G_a.

Definition 3.3. A subgraph K of G_a is said to be a *limb* of a if:

- a is a cutvertex of G_a and $K = G[C \cup \{a\}]$ where C is a connected component of $G_a - a$
- a is not a cutvertex of G_a and $K = G[C]$ where C is a connected component of $G_a - \text{Core}(a)$.

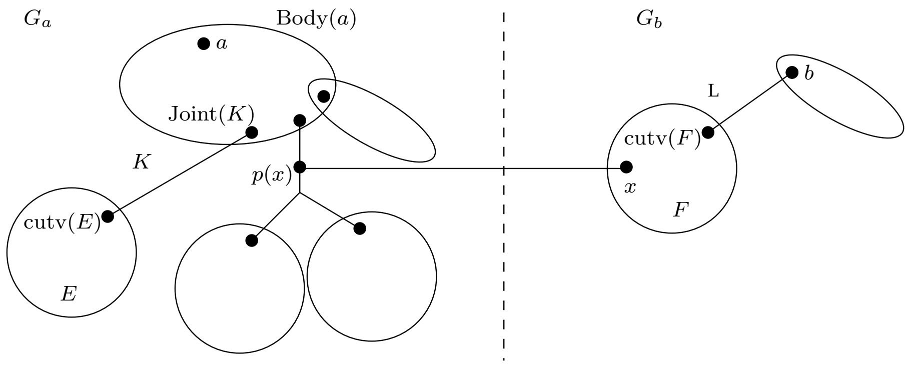

Figure 1: The diagram displays various parts of G_a and G_b. The broken line separates G_a and G_b. In G_a, $\mathrm{Body}(a) \neq \{a\}$ and a has three limbs. In G_b, b is a cutvertex and one of its limbs L contains an endblock F with exit vertex x.

The *joint* of a limb K, $\mathrm{Joint}(K)$, is the unique vertex $v \in K \cap \mathrm{Body}(a)$.

The reader may find it helpful to examine Figure 1. The circles and ellipses will always denote blocks in the block-cutvertex decomposition of the graph.

The proof of Theorem 2.3 will proceed by induction on d. The case $d = 2$ follows from Menger's theorem, as if G is 2-connected it contains two disjoint $a - b$ paths, one of which must have length at least 2. We will suppose for contradiction that the Theorem fails for some $d > 2$ and take G to be a minimal counterexample so that Theorem 2.3 holds for all smaller degrees and all graphs G' with $|G'| < |G|$. The following lemma will be the main step in the proof of Theorem 2.3. Its proof will be the aim of the next three sections.

Lemma 3.4. *Let G be a 2-connected subgraph of Q_n and $a, b \in V(G)$ such that $d(v) \geq d$ for all $v \in V(G) - \{a, b\}$, where $d \geq 3$. Suppose that Theorem 2.3 is true for smaller degrees and all graphs G' with $|G'| < |G|$. Suppose furthermore that there exists a splitting direction i for G_a and G_b in Lemma 2.2 for which $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$. Then G contains an $a - b$ path of length at least 2^{d-1}.*

Note that it follows from Lemma 3.4 that if $d_G(a) \geq 3$ and $d_G(b) \geq 3$, G contains an $a - b$ path of length at least 2^{d-1}. Indeed, taking any direction i on which a and b differ as the splitting direction in the proof of Lemma 2.2, we have $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$. Lemma 3.4 therefore applies and gives an $a - b$ path of length at least 2^{d-1}, as claimed.

Over the next three sections we will establish some results which will be used in the proof of Lemma 3.4 in Section 5. The first of these describes the block structure of G_a provided we cannot use endblock paths to form an $a - b$ path of length at least 2^{d-1}.

Lemma 3.5. *Suppose that G, a, b, G_a and G_b are as in the statement of Lemma 3.4. If G does not contain an $a - b$ path of length at least 2^{d-1} then the following hold:*

(i) *Every endblock of G_a which does not contain a in its interior must contain at least two exit vertices.*

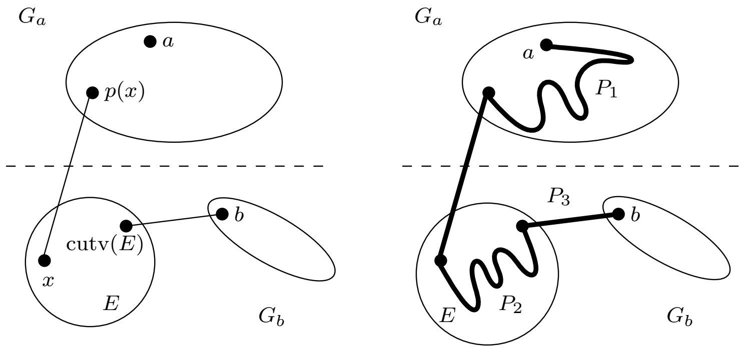

Figure 2: Path P constructed in Lemma 3.5(ii). Curved paths like P_1 and P_2 will represent endblock paths of length at least 2^{d-2} throughout.

(ii) *G_a is not 2-connected.*

(iii) *a does not lie in the interior of an endblock in G_a.*

(iv) *a must have at least two limbs.*

Proof. (i) Suppose not and let E be such an endblock. By the 2-connectivity of G, E must contain an exit vertex x. If x is its only exit vertex then every $v \in E - \{\text{cutv}(E), x\}$ has degree least d in E – such v must exist since $d \geq 3$. Since $|E| < |G|$, E contains a path P_2 of length at least 2^{d-1} from cutv(E) to x. Joining a to cutv(E) in G_a by a path P_1 and $p(x)$ to b in G_b by a path P_3 we have created a path $P_1P_2P_3$ of length at least 2^{d-1} from a to b, a contradiction.

(ii) Suppose G_a is 2-connected. First consider the case where G_b is not 2-connected. Let E be an endblock in G_b not containing b in its interior and take x to be an exit vertex of E with $p(x) \neq a$ – this exists by (i). Since Theorem 2.3 holds for $d-1$, there are paths P_1 in G_a from a to $p(x)$ and P_2 in E from x to cutv(E) both of length at least 2^{d-2}. Taking a path P_3 from cutv(E) to b in G_b we have constructed a path $P = P_1p(x)xP_2P_3$ from a to b of length at least 2^{d-1}, a contradiction.

If G_b is 2-connected, then the same proof as in (i) shows that G_b must contain two exit vertices, one of which, x, has $x \neq b$ and $p(x) \neq a$. Again as Theorem 2.3 holds for $d-1$, we obtain endblock paths from a to $p(x)$ in G_a and from x to b in G_b both of length at least 2^{d-2}. Joining the two with edge $xp(x)$, G again contains an $a-b$ path of length at least 2^{d-1}, a contradiction.

(iii) Suppose a lies in the interior of an endblock E of G_a. As $d_{G_a}(a) \geq 2$ E is 2-connected. As Theorem 2.3 holds for $d-1$, we have an endblock path P_1 from a to cutv(E) in E of length at least 2^{d-2}. From (ii) G_a is not 2-connected and so it contains a second endblock E', with an exit vertex x. Again since Theorem 2.3 holds for $d-1$, E' contains an endblock path P_3 from cutv(E') to x of length 2^{d-2}. Join cutv(E) to cutv(E') by a path P_2 in G_a and $p(x)$ to b by a path P_4 in G_b. Combining all of these paths we have a path $P_1P_2P_3xp(x)P_4$ from a to b of length at least 2^{d-1}, a contradiction.

(iv) This follows from (ii) and (iii) as if G_a is not 2-connected and a does not lie in the interior of any endblock, a must have at least two limbs. $\square$

Note that by symmetry of a and b, Lemma 3.5 also applies on replacing a

with b. The next proposition gives a simple case in which we can use endblock paths to build our path of length 2^{d-1} from a to b.

Proposition 3.6. *Let G, a, b, G_a and G_b be as in the statement of Lemma 3.4. Suppose G does not contain an $a-b$ path of length at least 2^{d-1}. Then for any exit vertex x of an endblock E of G_a, $p(x)$ can never lie in the interior of an endblock F of G_b.*

Proof. From Lemma 3.5(iii) $a \notin \text{int}(E)$ and $b \notin \text{int}(F)$. Pick a path P_1 in G_a from a to $\text{cutv}(E)$ and a path P_4 in G_b from $\text{cutv}(F)$ to b. Since E is 2-connected and all $v \in E - \{\text{cutv}(E), x\}$ have degree at least $d-1$ in $G[E]$, Theorem 2.3 gives a path P_2 of length at least 2^{d-2} from $\text{cutv}(E)$ to x. Similarly F contains a path P_3 of length at least 2^{d-2} from $p(x)$ to $\text{cutv}(F)$. Combining these gives an $a-b$ path $P = P_1P_2xp(x)P_3P_4$ of length at least 2^{d-1}, a contradiction. □

4 The Interaction Digraph

Throughout this section, G will be a 2-connected subgraph of Q_n containing vertices a and b, with $d_{G_a}(a) \geq 2$, $d_{G_a}(a) \geq 2$ and $d(v) \geq d$ for all $v \in V(G) - \{a, b\}$. We will also assume that Theorem 2.3 holds for all smaller degrees and for all graphs G' with $|G'| < |G|$.

Let $K_1, \ldots, K_r$ be the limbs of a and $L_1, \ldots, L_s$ be the limbs of b. Lemma 3.5(iv) shows that $r, s \geq 2$.

We form an auxiliary bipartite multidigraph $H = (A, B, \overrightarrow{E})$ which will represent the interaction between the limbs and cores of a and b. Let $A = \{K_1, \ldots, K_r\}$ and $B = \{L_1, \ldots, L_s\}$. Additionally, adjoin $\text{Core}(a)$ to A and $\text{Core}(b)$ to B if they are non-empty. Given an endblock E of G_a, there exists an exit vertex x with $x \neq a$ and $p(x) \neq b$ by Lemma 3.5(i) and (iii). Pick exactly one such exit vertex x_E for each such endblock E and adjoin a directed edge to H from K to $W \in B$ where E is contained in limb K and $p(x_E) \in W$. Similarly, for each endblock F in L we pick an exit vertex $y_F \in F$ with $p(y_F) \neq a$ and add a directed edge to H from L to V where $p(y_F) \in V$.

Note that by Proposition 3.6 we never choose an exit vertex x_E for some E and y_F for some F such that $p(x_E) = y_F$. Also, since any limb of a or b contains an endblock, every limb vertex in H must have outdegree at least one and core vertices have no outneighbours.

We shall study the component structure of H. The next two lemmas say that this is very restricted. Together they will allow us to find a connected component C of H consisting entirely of limbs. The inductive step in Section 5 will take place on the subgraph of G corresponding to this C.

As H is a multidigraph, we stress that in the next lemma, by a path we mean a path without repeated vertices.

Lemma 4.1. *Let G, a, b, G_a and G_b be as above. Suppose G does not contain an $a-b$ path of length at least 2^{d-1}. Then H does not contain an undirected path of length three.*

Proof. Suppose for contradiction that we have such a path $Q = V_0V_1V_2V_3$ in H and assume $V_0 \in A$. Each directed edge $\overrightarrow{VW}$ of Q gives an endblock in V with exit vertex x, such that $p(x) \neq b$ and $p(x) \in W$. These endblocks are distinct by the construction of H and since Theorem 2.3 holds for $d-1$, in each we can find an endblock path of length at least 2^{d-2} from its cutvertex to this exit vertex. We claim that we can form an $a-b$ path P which extends all three of these paths. As such a path has length at least $3(2^{d-2}) > 2^{d-1}$, this contradicts the hypothesis and proves the lemma.

We will construct our path by forming paths P_i in each V_i and eventually join them into one. The start point of P_i will be denoted by a_i and its end point by b_i. We first choose these vertices.

If $\overrightarrow{V_iV_{i+1}}$ is an edge of Q there is an endblock E in V_i with an exit vertex x_E such that $p(x_E) \in V_{i+1}$. In this case let $b_i = x_E$ and $a_{i+1} = p(x_E)$. If $\overleftarrow{V_iV_{i+1}}$ is an edge of Q this gives an endblock E in V_{i+1} with an exit vertex x_E such that $p(x_E) \in V_i$. In this case let $b_i = p(x_E)$ and $a_{i+1} = x_E$. We set

$$a_0 = \begin{cases} \text{Joint}(V_0) & \text{if } V_0 \text{ is a limb of } a; \\ b_0 & \text{if } V_0 = \text{Core}(a) \end{cases}$$

$$b_3 = \begin{cases} \text{Joint}(V_3) & \text{if } V_3 \text{ is a limb of } b; \\ a_3 & \text{if } V_3 = \text{Core}(b). \end{cases}$$

Note that b_i and a_{i+1} are adjacent for $i \in \{0,1,2\}$ and $a, b \notin \{b_0, a_1, b_1, a_2, b_2, a_3\}$.

We now build the paths P_i from a_i to b_i in each V_i, where V_i is a limb. We claim we can choose P_i so that neither a nor b are interior vertices of P_i (that is, they can lie on P_i, but only as end vertices) such that P_i has length at least 2^{d-2} if V_i has one outneighbour on Q and 2^{d-1} if V_i has two outneighbours on Q. Indeed, if V_i has exactly one outneighbour in Q then exactly one of a_i or b_i must be an exit vertex of an endblock E of V_i. Without loss of generality this is a_i. We must also have $b_i \notin \text{int}(E)$. Indeed, by definition b_3 never lies in the interior of an endblock, so $i \leq 2$ and a_{i+1} must be an exit vertex for a endblock in V_{i+1}. But as b_i and a_{i+1} are adjacent, this contradicts Propostion 3.6. Therefore, since Theorem 2.3 holds for $d-1$, E contains a path of length 2^{d-2} from a_i to the cutv(E). Since $V_i - \{a, b\}$ is connected for all i from the definition of a limb, we can extend this path from cutv(E) to b_i in V_i as required. The case where V_i has two outneighbours in Q is identical, using the same argument in two endblocks of V_i and joining their cutvertices in V_i.

Finally we combine the P_i paths. We first deal with the case where neither Core(a) nor Core(b) occur as interior vertices of Q. Combining the paths above we have an $a_0 - b_3$ path $P' = P_0b_0a_1P_1b_1a_2P_2b_2a_3P_3$. If Body$(a) = \{a\}$ then P' starts at a so we only need to extend P' to start at a when Body$(a) \neq \{a\}$. In P' as constructed above, Body$(a) \cap P'$ contains a_0 and at most one other vertex - indeed as the paths P_i above always lie entirely inside V_i, they can only intersect Body(a) in Joint(V_i) and therefore P' contains at most a_0 and Joint(V_2). Since Body(a) is 2-connected it contains a path P_1' from a to a_0 avoiding Joint(V_2). Finding a similar path P_2' from b_3 to b in Body(b) if Body$(b) \neq \{b\}$ we may take $P = P_1'P'P_2'$.

If Q contains one of the Core vertices, without loss of generality let it be Core(a). If Core(a) occurs as an interior vertex of Q, it must be V_2. Body(a) then contains distinct a_0, a_2, b_2 and we have two paths $P_1' = P_0b_0a_1P_1b_1a_2$ from a_0 to a_2 and $P_2' = b_2a_3P_3$ from b_2 to b_3 as in Figure 3. From the choice of the a_2 and b_2 above and the fact that a is not a cutvertex we have $a \notin \{a_0, a_2, b_2\}$. Therefore by 2-connectivity Body(a) contains two vertex disjoint paths from

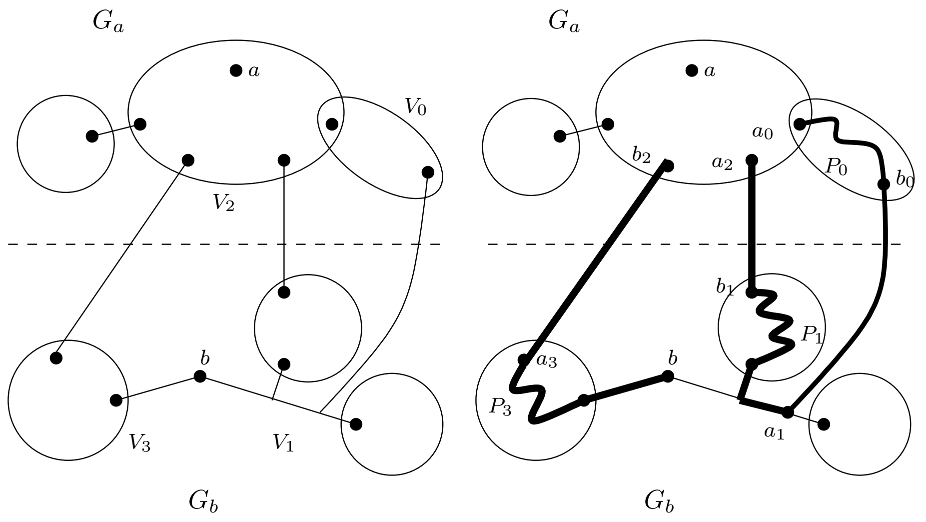

Figure 3: An illustration of Lemma 4.1 in the case where $V_2 = \text{Core}(a)$ and V_0V_1, V_1V_2 and V_3V_2 are directed edges of Q. As in the proof of Lemma 4.1, 2-connectivity can be used in Body(a) to find vertex disjoint paths from $\{a_0, a_2\}$ to $\{b, b_2\}$.

$\{a_0, a_2\}$ to $\{a, b_2\}$. Piecing these paths together with P_1' and P_2' we obtain an ab_3-path P'. If $\text{Body}(b) = \{b\}$ we are done since $b = b_3$. Otherwise we extend P' using 2-connectivity as above to find an $a - b$ path of length at least 2^{d-1}, contradicting the choice of G. □

Note that Lemma 4.1 guarantees that H has at least two connected components. The next lemma further limits H. Its proof is very similar to that of Lemma 4.1.

Lemma 4.2. *Let G, a, b, G_a and G_b be as above. Suppose that G does not contain an $a - b$ path of length at least 2^{d-1}. Furthermore, suppose that Body(a) $\neq \{a\}$. Then no component of H contains two vertices of A.*

Proof. Suppose H has such a component C. Then, since H does not contain a path of length three by Lemma 4.1, C consists of vertices $V_1, \ldots, V_t$ in A and a vertex W in B. At most one of $V_1, \ldots, V_t, W$ can be a core vertex as there is no edge between Core(a) and Core(b) in H.

If $W = \text{Core}(b)$ then V_1 and V_2 must be limbs and these guarantee two vertex disjoint paths P_1, P_2 from vertices $a_1, a_2 \in \text{Body}(a)$ to vertices $b_1, b_2 \in \text{Body}(b)$ both of length at least 2^{d-2} with $|P_i \cap \text{Body}(c)| = 1$ for $i = 1, 2$ and $c \in \{a, b\}$. As b has at least two limbs by Lemma 3.5(iv) and by Lemma 4.1 C cannot contain both of these, H must contain a second component C' containing a limb of b. This guarantees the existence of a third path P_3 from a vertex $a_3 \in \text{Body}(a)$ to $b_3 \in \text{Body}(b)$ of length 2^{d-2} again with $|P_3 \cap \text{Body}(c)| = 1$ for $c \in \{a, b\}$ which is disjoint from P_1 and P_2. Using identical 2-connectivity arguments in both Body(a) and Body(b) as in Lemma 4.1 we can combine these three paths into one from a to b, contradicting the hypothesis.

If $W \neq \text{Core}(b)$ then C guarantees a path P_1 of length 2^{d-1} between two vertices a_1 and a_2 in Body(a) with $|P_1 \cap \text{Body}(a)| = 2$, $b \notin P_1 \cap \text{Body}(b)$

and $|P_1 \cap \text{Body}(b)| \leq 1$. Again from a second connected component of H we obtain a disjoint path P_2 from an element $a_3 \in \text{Body}(a)$ to $b_1 \in \text{Body}(b)$ with $|P_2 \cap \text{Body}(c)| \leq 1$ for $c \in \{a, b\}$. Once more, with an application of 2-connectivity in $\text{Body}(a)$ and a possible application in $\text{Body}(b)$ we find an $a - b$ path extending both P_1 and P_2, a contradiction. $\square$

Again the same applies switching a with b. As mentioned before Lemma 4.1 the previous two lemmas imply that H contains a connected component C consisting entirely of limbs.

Corollary 4.3. *Let G, a, b, G_a and G_b be as above. Suppose that G does not contain an $a - b$ path of length 2^{d-1}. Then the interaction digraph H of G has at least two connected components, one of which C consists entirely of limbs.*

Proof. Since $|A|, |B| \geq 2$, if H is connected it contains an undirected path of length three, contradicting Lemma 4.1. Therefore H has at least two connected components, as claimed. If H does not contain a component consisting entirely of limbs, each component of H contains one of $\text{Core}(a)$ or $\text{Core}(b)$. But then H has exactly two connected components, one containing $\text{Core}(a)$ and one containing $\text{Core}(b)$. But as A contains $\text{Core}(a)$ and at least two limbs, two of these must lie in the same connected component of H, contradicting Lemma 4.2. $\square$

We will write G_C for the subgraph $G[\cup_{W \in C} V(W)]$ of G. We note that G_C must contain exactly one vertex a_C in $\text{Body}(a)$ and one vertex b_C in $\text{Body}(b)$ – if $\text{Body}(a) = \{a\}$ then $a_C = a$, if not then by Lemma 4.2 $A \cap C = \{V\}$ and we may take $a_C = \text{Joint}(V)$.

5 The Inductive Step

Suppose that G, a, b, G_a and G_b satisfy the hypothesis of Lemma 3.4 but G does not contain a path of length 2^{d-1}. Then we may apply Corollary 4.3 to find a component C of H consisting entirely of limbs. Our final lemma before the proof of Lemma 3.4 finds a subgraph of G_C which either also satisfies the conditions of Theorem 2.3 or builds half of the $a - b$ path we are looking for from any edge entering it. Before stating it we give one last definition.

Definition 5.1. Given a graph G and $S \subset V(G)$ define the $\text{span}_G(S)$ to be the subset of $V(G)$ consisting of all vertices which lie on a path between two elements of S.

Note that we include paths of length zero in this definition, so that $S \subset \text{span}_G(S)$.

Lemma 5.2. *Let G be a 2-connected subgraph of Q_n containing vertices a and b such that $d_{G_a}(a) \geq 2$, $d_{G_b}(b) \geq 2$ and $d(v) \geq d$ for all $v \in V(G) - \{a, b\}$. Suppose that Theorem 2.3 holds for all smaller degrees and for all graphs G' with $|G'| < |G|$. Suppose furthermore that G does not contain an $a - b$ path of length at least 2^{d-1}. Then taking C as in Corollary 4.3, G_C has a 2-connected subgraph J containing two vertices $a' \in G_a$ and $b' \in G_b$ with the following properties:*

(i) *every vertex $v \in J - \{a', b'\}$ has degree at least $d - 1$ in J and all the neighbours of v in G_C are contained in J.*

(ii) *for any vertex* $v \in J - \{a', b'\}$*,* J *contains an* $a' - v$ *path not containing* b *and a* $b' - v$ *path not containing* a*, both of length at least* 2^{d-2}*.*

Proof. From Lemma 4.1, C cannot contain two limbs of both a and b. We may therefore assume that C consists of one limb K of a and limbs $L_1, \ldots, L_t$ of b.

For each $i \in [t]$ we define $S_i \subset K$ and $T_i \subset L_i$ as follows:

$$S_i := \{v \in K : v \text{ has a neighbour } p(v) \in L_i - \{b\}\}$$

and

$$T_i := \{w \in L_i : w \text{ has a neighbour } p(w) \in K - \{a\}\}.$$

Now each limb in the interaction digraph has at least one outneighbour. We claim that for each endblock $E \in K$ there exists some $i \in [t]$ with $|S_i| \geq 2$ such that $\mathrm{int}(E) \cap S_i \neq \emptyset$. Indeed, from construction of the interaction graph, E contributes a directed edge from K to L_i for some $i \in [t]$. This gives an exit vertex $x_E \in \mathrm{int}(E)$ with $p(x_E) \in L_i - \{b\}$. Similarly we have an exit vertex y of an endblock in L_i with $p(y) \in K - \{a\}$. Now by Proposition 3.6 we have $x_E \neq p(y)$ and both are contained in S_i, proving the claim.

Assume that $L_1, \ldots, L_t$ are labelled so that for $i \in [1, t']$, $|S_i| \geq 2$ and $|S_i| = 1$ for $i \in [t'+1, t]$. By the previous paragraph we have $t' \geq 1$. For all $I \subset [t]$ we let $S_I = \bigcup_{i \in I} S_i$.

Beginning with the $\{S_1, \ldots, S_{t'}\}$, repeatedly replace sets S_I and S_J in this list with $S_{I \cup J}$ if $|\mathrm{span}_{G_a}(S_I) \cap \mathrm{span}_{G_a}(S_J)| \geq 2$. When this proceedure ends we are left with sets $\{S_{I_1}, \ldots, S_{I_p}\}$.

Now clearly $\mathrm{span}_{G_a}(S_{I_l})$ is a union of blocks of K for all $l \in [p]$ and by our construction proceedure above, no two can share a block. Also from the claim above, each endblock E of K is contained in $\mathrm{span}_{G_a}(S_{I_l})$ for some $l \in [p]$. Combining these two facts it is easy to see that there is some $l \in [p]$ for which $\mathrm{span}_{G_a}(S_{I_l})$ is separated from $G_a - \mathrm{span}_{G_a}(S_{I_l})$ in G_a by a single vertex a'.

We are now ready to choose J. Let $N \subset [t'+1, t]$ consist of all n for which $S_n \cap (\mathrm{span}_{G_a}(S_{I_l}) - \{a'\}) \neq \emptyset$ and let $M = I_l \cup N$. We take $J = G[\mathrm{span}_{G_a}(S_M) \cup \mathrm{span}_{G_b}(T_M)]$.

Lastly we choose b'. We pick this vertex depending on whether $|M| = 1$ or $|M| \geq 2$. If $|M| = 1$ then $I_l = \{i\}$ for some $i \in [t']$ and $N = \emptyset$. Since $G_b[\mathrm{span}_{G_b}(T_i)]$ is a connected union of blocks of L_i and contains a vertex in the interior of every endblock of L_i, this graph must be separated from $G_b - G_b[\mathrm{span}_{G_b}(T_i)]$ by a single vertex b' in G_b.

If $|M| \geq 2$ then as C contains at least two limbs of b, by Lemma 4.2 $\mathrm{Body}(b) = \{b\}$. Then $J = G[\mathrm{span}_{G_a}(S_M) \cup (\bigcup_{j \in M} V(L_j))]$ and we may take $b' = b$.

It is clear from construction that J satisfies (i). It is also easy to show that J is 2-connected. Indeed, suppose we remove $v \in J \cap G_a$. Given any vertex $w \in J \cap G_a - \{v\}$, $w \in G[\mathrm{span}_{G_a}(S_{I_l})]$ so either $w \in S_{I_l}$ or w lies on a path between two elements of S_{I_l}. In both cases there exists a path from w to $J \cap G_b$. Since $J \cap G_b$ is connected this shows that $J - v$ is connected for all $v \in J \cap G_a$. A similar argument shows that $J - v$ is connected for all $v \in J \cap G_b$.

We now show that (ii) holds for J. Suppose that $v \in J - \{a', b'\}$ and that we are looking for an $a' - v$ path not containing b of length at least 2^{d-2}. We claim the following:

Claim 1: $J - b$ contains a 2-connected subgraph J' containing all of $J \cap G_a$.

If $b \notin J$ then this is immediate taking $J' = J$, so we may assume that $b = b' \in J$. It suffices to show that there exists such a subgraph J' of $G[\mathrm{span}_{G_a}(S_{I_l}) \cup \mathrm{span}_{G_b}(T_{I_l})]$.

For each $i \in [t']$, $G[\mathrm{span}_{G_a}(S_i) \cup \mathrm{span}_{G_b}(T_i)] - \{b\}$ has a 2-connected subgraph which contains all of $\mathrm{span}_{G_a}(S_i)$, namely $G[\mathrm{span}_{G_a}(S_i) \cup \mathrm{span}_{G_b}(p(S_i))] - \{b\}$ where $p(S_i) := \{p(s) : s \in S_i\}$. Here $|S_i| \geq 2$ guarantees that this graph is 2-connected. Now since the union of two 2-connected graphs which intersect at least two points is still a 2-connected graph, from the joining proceedure which produced the set I_l we must have that the union of $G[\mathrm{span}_{G_a}(S_i) \cup \mathrm{span}_{G_b}(p(S_i))] - \{b\}$ for $i \in I_l$ is a 2-connected graph J'. Moreover, this graph clearly contains all of $G[\mathrm{span}_{G_a}(S_{I_l})] = J \cap G_a$. This proves the claim.

We now use J' to find the $a' - v$ path claimed in (ii). First find a path P_1 from v to some $w \in J' - a'$ which avoids b. Such a path is immediate if $v \in J'$, so we may assume $v \notin J'$. Let $v \in L_i$, $i \in M$. If $L_i - b$ contains some element of J', take P_1 to be the shortest path in $L_i - b$ from v to an element w in J'. If not, we take P_1' to be a path in $L_i - b$ to an exit vertex x_F of some endblock F of L_i, $p(x_F) = w$ and $P_1 = P_1' x_F p(x_F)$. Note that in both these cases P_1 intersects J' only in one vertex $w \neq a'$.

Now take an endblock E of $J \cap G_a$. If $w \in \mathrm{int}(E)$ then G_a contains a path from w to a' which extends an endblock path in E. As such a path has length at least 2^{d-2} we can assume $w \notin \mathrm{int}(E)$. Let J'' denote the graph J' with $\mathrm{int}(E)$ contracted to a single vertex e. It is easy to see that this graph is still 2-connected. Therefore there exists two vertex disjoint paths P_2 and P_3 from $\{a', w\}$ to $\{\mathrm{cutv}(E), e\}$. Say that these paths are P_2 from a' to $\mathrm{cutv}(E)$ and P_3 from w to e. If $P_3 = P_3' xe$, x must have a partner $p(x) \in \mathrm{int}(E)$. This gives a path $P_3' x p(x)$ from w to $p(x)$ in J. Now since E is a 2-connected graph and for all $v \in E - \{\mathrm{cutv}(E), p(x)\}$ $d_E(v) \geq d - 1$, we can apply Theorem 2.3 to E to find a $p(x) - \mathrm{cutv}(E)$ path P_4 of length at least 2^{d-2}. Combining all of these paths gives a $v - a'$ path $P = P_1 P_3' x p(x) P_4 P_2^{(r)}$ of length at least 2^{d-2}, where $P_2^{(r)}$ is P_2 reversed.

An identical argument gives the $b' - v$ path claimed in (ii). $\square$

We are now ready to prove Lemma 3.4.

Proof of Lemma 3.4. Suppose for contradiction that G does not contain an $a - b$ path of length at least 2^{d-1}. Then by Corollary 4.3, the interaction digraph H of G must contain a connected component C consisting entirely of limbs.

As G does not contain an $a - b$ path of length at least 2^{d-1}, we can apply Lemma 5.2 to find a 2-connected subgraph J of G_C and vertices a' and b' which satisfy Lemma 5.2 (i) and (ii). Note that $|J| < |G|$ since H contains two connected components and J is contained entirely in one of them.

Now if there are no edges between $J - \{a', b'\}$ and $G - J$, all $v \in J - \{a', b'\}$ have degree at least d in J. But then since $|J| < |G|$, J contains an $a' - b'$ path P of length at least 2^{d-1}. Extending this path from a' to a and from b' to b gives an $a - b$ path of length at least 2^{d-1}, a contradiction. Therefore such an edge must exist, joining say $v \in J - \{a', b'\}$ to $w \in G - J$. By Lemma 5.2 (i) $w \notin G_C$. We may assume $w \in G_a$.

Suppose first that $w \in \mathrm{Body}(a)$. Here $\mathrm{Body}(a) \neq \{a\}$ by Lemma 5.2 (i). Take an $a' - v$ path of length 2^{d-2} in J as guaranteed by Lemma 5.2 (ii) which does not contain b. This path extends in G_C to an $a_C - v$ path P_1, where

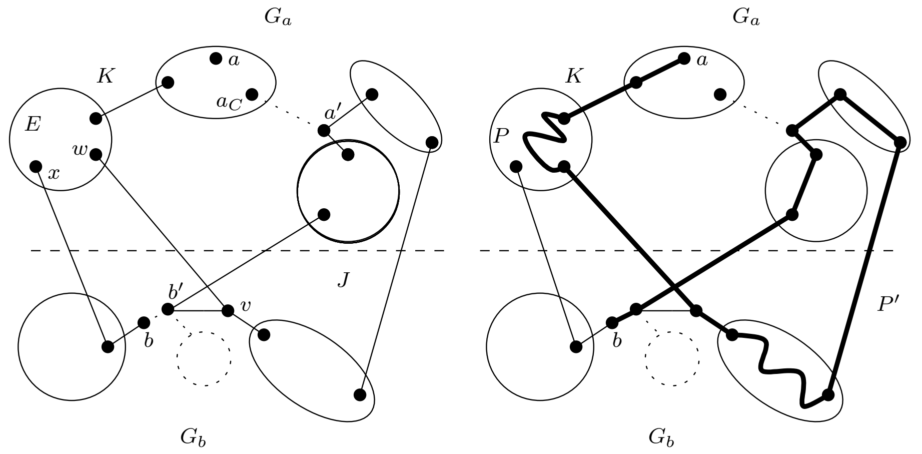

Figure 4: An illustration of the case $w \in \text{int}(E)$ in the proof of Lemma 3.4. The broken dotted pieces represent vertices in G_C that are left out of J.

$a_C = G_C \cap \text{Body}(a)$. As this path lies entirely in G_C, it can only intersect $\text{Body}(a)$ in a_C and $\text{Body}(b)$ in at most $b_C = G_C \cap \text{Body}(b)$. Pick a limb K of a not contained in C and an endblock E of K. K contains a path of length 2^{d-2} from $\text{Joint}(K)$ to the exit vertex x_E of E. As $p(x_E) \notin G_C$, we can find a path P_2 from $p(x_E)$ to b in G_b disjoint from P_1. But now since $\text{Body}(a)$ is 2-connected we can find vertex disjoint paths from $\{a, \text{Joint}(K)\}$ to $\{a_C, w\}$. Combining these paths with P_1 and P_2 we obtain a path of length at least $2^{d-1} + 2$ from a to b, again a contradiction.

Therefore we can assume $w \in K$ for some limb K of a not in C. If $w \notin \text{int}(E)$ for some endblock E of K then we can proceed exactly as in the case $w \in \text{Body}(a)$ above to find an $a - b$ path of length at least 2^{d-1}, so we may assume $w \in \text{int}(E)$. Then K contains a path of length at least 2^{d-2} from w to $\text{Joint}(K)$. Joining this path to the $v - b'$ path guaranteed by Lemma 5.2 (ii) via the edge wv we obtain a $\text{Joint}(K) - b'$ path of length at least $2^{d-1} + 1$. Extending this path from $\text{Joint}(K)$ to a and from b' to b we again find an $a - b$ path of length at least $2^{d-1} + 1$. This contradicts our assumption and proves the Lemma. $\square$

6 Removing the Degree Assumption

Again, let G be a 2-connected subgraph of Q_n with $a, b \in G$ such that $d_G(v) \geq d$ for all $v \in V(G) - \{a, b\}$. Also, suppose that Theorem 2.3 holds for smaller degrees and for all graphs G' with $|G'| < |G|$.

If we could find a splitting direction i for G_a and G_b in Lemma 2.2 so that $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$ then using Lemma 3.4 G would contain an $a - b$ path of desired length. This is possible if $d_G(a) \geq 3$ and $d_G(b) \geq 3$, so we may assume that say $d_G(a) = 2$. However, we can not guarantee this in general – for example, a and b could be adjacent with both having only one other neighbour in G.

Now the condition $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$ in previous sections ensured that all endblocks of G_a and G_b contained long paths, which is false if say a has a single neighbour a' in G_a. This fact was then used in Lemma 3.5 (iv) to show that a has at least two limbs in G_a which in turn was crucially used numerous times in our analysis of H e.g. Lemma 4.2.

In this section, we will extend the arguments from the proof of Lemma 3.4 to prove Theorem 2.3. The first step is the following lemma.

Lemma 6.1. *Let G be a 2-connected subgraph of Q_n with $a, b \in G$ such that $d_G(a) = 2$ and $d_G(v) \geq d$ for all $v \in V(G) - \{a, b\}$, where $d \geq 3$. Suppose that Theorem 2.3 holds for smaller degrees and for all graphs G' with $|G'| < |G|$. Furthermore, suppose that G does not contain an $a - b$ path of length at least 2^{d-1}. Then the following hold:*

(i) *$G - a$ is a 2-connected graph*

(ii) *There is a splitting direction i for G_a and G_b so that $d_{G_a}(a) \geq 2$ or $d_{G_b}(b) \geq 2$.*

Proof. (i) If $G - a$ is not 2-connected, it has at least two endblocks in its block-cutvertex decomposition, one of which E has $b \notin \mathrm{int}(E)$. Now since G is 2-connected, a must be joined to the interior of all the endblocks of $G - a$. As $d_G(a) = 2$, $G - a$ has exactly two endblocks with a having exactly one neighbour in the interior of each. Let w be this neighbour in E.

Now E is 2-connected (as $d \geq 3$) and all vertices in $E - \{\mathrm{cutv}(E), w\}$ have degree at least d in E. Since $|E| < |G|$, Theorem 2.3 applies to give a path P of length at least 2^{d-1} from w to $\mathrm{cutv}(E)$. Extending this path on either side to a and b respectively, we have an $a - b$ path of length at least 2^{d-1}, a contradiction.

(ii) We can always choose such a direction if a and b are at Hamming distance at least three in Q_n or if one of a or b have degree greater than 2 in G. So we can assume a and b are at Hamming distance one or two and both have degree exactly two in G.

First consider a and b at Hamming distance one. If they are not adjacent in G we can choose the direction on which they differ for i so we can assume they are adjacent. Then a and b both have one other neighbour in G, a' and b' respectively. Now if $G - \{a, b\}$ is 2-connected we can apply Theorem 2.3 to $G - \{a, b\}$ with a' and b' in place of a and b. This gives an $a' - b'$ path of length at least 2^{d-1}. Adjoining the edges aa' and bb' to this path we have an $a - b$ path of length $2^{d-1} + 2$, more than enough. If $G - \{a, b\}$ is not 2-connected it is easily seen that a' and b' must lie in the interior of different endblocks of $G - \{a, b\}$. We can therefore find a path from a' to b' in $G - \{a, b\}$ which extends two endblock paths. Adjoining the edges aa' and bb' to this path, we have an $a - b$ path of length at least $2^d + 2$, a contradiction.

If a and b are at Hamming distance two, we can always find such a direction i unless a and b are joined to the same two neighbours in G, a' and b' say. Then $\{a, a', b, b'\}$ form a C_4 with a opposite b. Working with $G - \{a, b\}$, a' and b' as above, we again obtain an $a - b$ path of desired length in G. □

We are now ready for the proof of Theorem 2.3.

Proof of Theorem 2.3. The proof is by induction on d and $|G|$. The base case $d = 2$ follows from Menger's theorem, as if G is 2-connected it contains two disjoint $a - b$ paths, one of which must have length at least 2.

Suppose that G is as in the statement of the Theorem and that Theorem 2.3 holds for all smaller values of d and for all graphs G' with $|G'| < |G|$. Suppose for contradiction that G does not contain a $a - b$ path of length at least 2^{d-1}. By Lemma 3.4, we must have that for all choices of i in Lemma 2.2 either $d_{G_a}(a) \leq 1$ or $d_{G_b}(b) \leq 1$.

Take the splitting direction i for G_a and G_b as in Lemma 6.1(ii). We will assume without loss of generality that $d_{G_b}(b) \geq 2$. By the previous paragraph we must have $d_{G_a}(a) = 1$. Let the neighbours of a be $a' \in G_a$ and $v \in G_b$.

Now we can assume $v \neq b$. Otherwise, by Lemma 6.1(i), $G - a$ is 2-connected and as all $v \in V(G - a) - \{a', b\}$ have degree at least d in $G - a$, by induction, $G - a$ contains an $a' - b$ path of length at least 2^{d-1}. Appending the edge aa' to this path we obtain an $a - b$ path of length at least 2^{d-1}, a contradiction.

Lemma 3.5 (i)-(iv) still hold for G_b with the same proofs as before. In particular, b still has at least two limbs. We make the following claim:

Claim 2: $v \in \mathrm{int}(E_v)$ for some endblock E_v of G_b

Suppose otherwise. From Lemma 3.5(iii) b does not lie in the interior of an endblock of G_b and by Lemma 3.5(iv) G_b contains two vertex disjoint paths P_1 from v to $\mathrm{cutv}(E_1)$ and P_5 from $\mathrm{cutv}(E_2)$ to b, where E_1 and E_2 are two endblocks of G_b. Taking exit vertices x_1 and x_2 of E_1 and E_2 respectively, by induction on d, E_1 contains a path P_2 of length at least 2^{d-2} from $\mathrm{cutv}(E_1)$ to x_1 and E_2 contains a path P_4 of length at least 2^{d-2} from x_2 to $\mathrm{cutv}(E_2)$. Taking a path P_3 from $p(x_1)$ to $p(x_2)$ in $G_a - a$ and combining the paths, G contains an $a - b$ path $avP_1P_2x_1p(x_1)P_3p(x_2)x_2P_4P_5$ of length at least 2^{d-1}. This contradicts our assumption and proves the claim.

We now again construct an interaction digraph H but this time it is built from the limbs of a' and b instead of those of a and b. Note that $\{a, a'\}$ is a limb of a' and so, both a' and b have at least two limbs. Take $H = (A', B, \overrightarrow{E})$ to be a bipartite multidigraph on vertex sets $A' = \{K_1, \ldots, K_r\}$ and $B = \{L_1, \ldots L_s\}$, the set of limbs of a' and b respectively. We also adjoin $\mathrm{Core}(b)$ to B if it is non-empty ($\mathrm{Core}(a') = \emptyset$ since a' is a cutvertex of G_a). Now each endblock of G_a other than $\{a, a'\}$ contains at least two exit vertices, as in Lemma 3.5(i). Therefore for each endblock E of G_a or G_b other than $\{a, a'\}$ we can pick an exit vertex x_E with $p(x_E) \neq a', b$. From our claim above we can pick $x_{E_v} = v$. Now adjoin a directed edge from $K \in A'$ to $L \in B$ for each endblock E in K with $p(x_E) \in L$ and a directed edge from $L \in B$ to $K \in A'$ for each endblock E in L with $p(x_E) \in K$. Note that every limb other than $\{a, a'\}$ still has an outneighbour in H.

For this H Lemma 4.1 and Lemma 4.2 still hold, again with the same proofs as before. Using these two, as in Corollary 4.3, we can show that H contains a connected component C consisting entirely of limbs which does not contain the limb $\{a, a'\}$. Indeed, b has at least two limbs so pick one, $L \in B$, not containing E_v and take C to be the connected component of H containing L. As v is the unique neighbour of a in G_b and $v \notin L$, if $\{a, a'\} \in C$ then H would contain a path of length three, contradicting Lemma 4.1. Furthermore, since $\mathrm{Core}(a') = \emptyset$, if C did not consist entirely of limbs of a' and b, $\mathrm{Body}(b) \neq \{b\}$ and C contains two vertices of B, contradicting Lemma 4.2.

The remainder of the proof of Theorem 2.3 is almost identical to that of Lemma 3.4. We can apply Lemma 5.2 to find a subgraph J of G_C. Using this subgraph as in the proof of Theorem 3.4 we either obtain an $a'-b$ path of length at least 2^{d-1} which is contained entirely in G_C or an $a'-b$ path of length at least $2^{d-1}+1$ in G. In the first case we find our $a-b$ path by appending the edge $a'a$ the to $a'-b$ path. In the second case, unless a is already a vertex of this path we can also do this. But if a a vertex of this $a'-b$ path, it must occur as the second vertex. Deleting a' from the path, we obtain an $a-b$ path of length at least 2^{d-1}, as required. □

7 A Tight Bound

In this section we will prove Theorem 2.1. Its proof has the same structure as Theorem 2.3 but requires more care in various arguments. The proof will again be by induction on d. The base case $d=2$ is immediate unless a and b are at Hamming distance 2 apart. If this is the case and G is not isomorphic to Q_2 pick any vertex v of G not in the unique 2-cube containing a and b. By 2-connectivity G contains vertex disjoint $a-v$ and $v-b$ paths, which when combined give a path of length at least 3, as required.

We will suppose that the Theorem fails for some $d>2$ and take G to be a minimal counterexample so that Theorem 2.1 holds for all smaller degrees and all graphs G' with $|G'|<|G|$. To begin we will prove the analogue of Lemma 3.4.

Lemma 7.1. *Let G be a 2-connected subgraph of Q_n, not isomorphic to Q_d and let $a,b \in V(G)$ such that $d(v) \geq d$ for all $v \in V(G)-\{a,b\}$, where $d \geq 3$. Suppose that Theorem 2.1 is true for smaller degrees and all graphs G' with $|G'|<|G|$. Suppose furthermore that there exists a splitting direction i for G_a and G_b in Lemma 2.2 for which $d_{G_a}(a) \geq 2$ and $d_{G_b}(b) \geq 2$. Then G contains an $a-b$ path of length at least 2^d-1.*

Our first step in the proof of Lemma 7.1 is to establish the analogue of Lemma 3.5.

Lemma 7.2. *Let G,a,b,G_a and G_b be as in the statement of Lemma 7.1. If G does not contain an $a-b$ path of length at least 2^d-1 the following hold:*

(i) *Every endblock E of G_a which does not contain a in its interior contains at least two exit vertices x and x'. Furthermore, we can choose these so that E contains cutv$(E)-x$ and cutv$(E)-x'$ paths of length at least $2^{d-1}-1$.*

(ii) *G_a is not 2-connected.*

(iii) *a does not lie in the interior of an endblock in G_a.*

(iv) *a must have at least two limbs.*

Proof. (i) Here the proof of Lemma 3.5(i) needs only a small change. If E is isomorphic to Q_{d-1} we can choose any two neighbours of $\text{cutv}(E)$ for x and x', so we may assume E is not isomorphic to Q_{d-1}. Now G is 2-connected so E contains at least one exit vertex x. Suppose for contradiction that this is the only one. Then as E is 2-connected, not isomorphic to Q_{d-1} with $d_E(v) \geq d$ for all $v \in E-\{\text{cutv}(E),x\}$ and $|E|<|G|$, it contains a $\text{cutv}(E)-x$ path of length at least 2^d-1. Extending this path as before we obtain an $a-b$ path of length at least $2^d - 1$, a contradiction. Therefore E has two exit vertices and as E is not isomorphic to Q_{d-1}, E contains paths of length at least $2^{d-1} - 1$ from $\mathrm{cutv}(E)$ to both of them.

(ii) The change to the proof of the Lemma 3.5 (ii) in this case is a little more involved. Suppose for contradiction that G_a is 2-connected.

First suppose that G_b is not 2-connected. If there exists an endblock E in G_b such that $b \notin E$, we have a path P_1 in G_b of length at least 1 from b to $\mathrm{cutv}(E)$. Pick an exit vertex x of E such that E contains a $\mathrm{cutv}(E){-}x$ path P_2 of length at least $2^{d-1} - 1$ and $p(x) \neq a$ – this exists by (i). Combining the paths P_1 and P_2 above with the $a{-}p(x)$ path of length $2^{d-1}{-}2$ in G_a guaranteed by Theorem 2.1, G contains an $a{-}b$ path of length at least $1{+}(2^{d-1}{-}1){+}1{+}(2^{d-1}{-}2) = 2^d{-}1$, a contradiction. So if G_b is not 2-connected b must lie in *every* endblock $E_1, \ldots, E_t$ of G_b. Note that since $t \geq 2$ this implies $b \notin \mathrm{int}(E_i)$ for any i.

Now using (i) as with E above, E_1 must have an exit vertex x such that E_1 contains a $x - b$ path of length at least $2^{d-1} - 1$, with $p(x) \neq a$. If G_a were not isomorphic to Q_{d-1}, it contains a path of length $2^{d-1} - 1$ from a to $p(x)$. Combining these two with the edge $xp(x)$ we obtain an $a - b$ path of length $2^d - 1$.

Therefore we can assume G_a is isomorphic to Q_{d-1}. Then G_a contains a path P_3 of length at least $2^{d-1} - 1$ from a to any of its neighbours. Take a neigbhour x such that $p(x) \neq b$. Here $p(x)$ must be in $\mathrm{int}(E_i)$ for some $i \in [t]$. Now $t \geq 2$ so E_i is not isomorphic to Q_{d-1} – otherwise G_a would receive too many edges from G_b by (i) above. Since Theorem 2.1 holds for smaller degrees, E_i contains a path P_4 from $b = \mathrm{cutv}(E_i)$ to $p(x)$ of length at least $2^{d-1} - 1$. Combining P_3 and P_4 with the edge $xp(x)$ we have an $a - b$ path of length at least $2^d - 1$, a contradiction.

The case when G_b is 2-connected is very similar. We can obtain two paths of length at least $2^{d-1} - 1$ in G_a and $G_b = E_1$ if neither of the two are isomorphic to Q_{d-1} and if one is isomorphic to Q_{d-1} we can use the same argument as in the case where G_a is isomorphic to Q_{d-1} and $t \geq 2$ above.

(iii) This is similar to (ii) but a little easier. Suppose that a is contained in the interior of some endblock E of G_a. As Theorem 2.1 holds for degrees smaller than d, E contains an $a - \mathrm{cutv}(E)$ path P_1 of length at least $2^{d-1} - 2$. Since G_a is not 2-connected by (ii), it also contains a second endblock E'. Now E' must contain an exit vertex x with $p(x) \neq b$ for which E' contains a $\mathrm{cutv}(E') - x$ path P_3 of length at least $2^{d-1} - 1$. Joining $\mathrm{cutv}(E)$ to $\mathrm{cutv}(E')$ by a path P_2 in G_a and $p(x)$ to b with a path P_4 in G_b gives an $a - b$ path $P = P_1 P_2 P_3 x p(x) P_4$ of length at least $(2^{d-1} - 2) + 0 + (2^{d-1} - 1) + 1 + 1 = 2^d - 1$, a contradiction.

(iv) Again follows from (ii) and (iii) as in Lemma 3.5(iv). □

The above modifications demonstrate the main problem in moving from the bounds of Theorem 2.3 to bounds of Theorem 2.1 – on combining endblock paths together without any care as before, we are usually left short a small number of vertices. While in the above Lemma we were able to exploit some small extremal arguments to obtain these extra vertices, such arguments do not allow us to prove the natural analogue of Proposition 3.6. Indeed, we may have an endblock E of G_a not isomorphic to Q_{d-1} and an endblock F of G_b isomorphic to Q_{d-1} with $\mathrm{cutv}(E) = a$ and $\mathrm{cutv}(F) = b$. If there is a vertex $x \in \mathrm{int}(E)$ at odd Hamming distance from a adjacent to a vertex $y \in \mathrm{int}(F)$ at even Hamming distance from b, our induction hypothesis only allows us to find a path of length at least $(2^{d-1}-1)+1+(2^{d-1}-2) = 2^d - 2$ from a to b.

Now Lemma 3.6 was important in the proof Theorem 2.3. In particular, it was used in the construction of the interaction digraph of G and crucially in the proof of Lemma 5.2 where it guaranteed the existence of the 2-connected subgraph J. The next proposition is a weakened version of Proposition 3.6 which will play a similar role in the proof of Theorem 2.1.

From Lemma 7.2(i) above, for every endblock E from G_a we can pick an exit vertex x_E so that E contains a $\text{cutv}(E) - x_E$ path of length at least $2^{d-1}-1$ and for which $p(x_E) \notin \{a, b\}$. Similarly pick such exit vertices y_F for endblocks F in G_b.

Proposition 7.3. *Let G, a, b, G_a and G_b be as in the statement of Lemma 7.1. Suppose G does not contain an $a-b$ path of length at least 2^d-1. Then for every endblock E of G_a, $p(x_E) \neq y_F$ for any endblock F of G_b. Furthermore, if Body$(a) \neq \{a\}$ or Body$(b) \neq \{b\}$ we also have $p(x_E) \notin$ int(F).*

Proof. From Lemma 7.2(iii) $a \notin \text{int}(E)$ and $b \notin \text{int}(F)$. Suppose for contradiction that $p(x_E) = y_F$. Combining the $\text{cutv}(E)-x_E$ path in E with the $\text{cutv}(F)-y_F$ path in F guaranteed by Lemma 7.2(i) via the edge $x_E p(x_E) = x_E y_F$, we have a $\text{cutv}(E)-\text{cutv}(F)$ path of length at least $(2^{d-1}-1)+1+(2^{d-1}-1) = 2^d-1$. As this path extends to an $a-b$ path, we have a contradiction.

The second part is similar. Since F is 2-connected and we have $d_F(v) \geq d-1$ for all $v \in F - \{\text{cutv}(F), p(x_E)\}$, F contains a $p(x_E) - \text{cutv}(F)$ path of length at least $2^{d-1}-2$. Combining this with the $\text{cutv}(E) - x_E$ path of length at least $2^{d-1}-1$ in E via the edge $x_E p(x_E)$ we have a $\text{cutv}(E) - \text{cutv}(F)$ path of length at least 2^d-2. If $\text{Body}(a) \neq \{a\}$ or $\text{Body}(b) \neq \{b\}$, extending this path to an $a-b$ path takes at least one more edge, again giving an $a-b$ path of length at least 2^d-1, a contradiction. □

We now look towards an slightly altered construction for H. Let our interaction digraph $H = \{A, B, \overrightarrow{E}\}$ again be a bipartite multidigraph whose bipartition consists of the limbs of a and b respectively. Again we additionally adjoin $\text{Core}(a)$ and $\text{Core}(b)$ to A and B respectively if they are non-empty. For each endblock E of G_a, adjoin a directed edge to H from $K \in A$ to $L \in B$ if E is an endblock of K and $p(x_E) \in L$. Similarly, for each endblock F of G_b, adjoin a directed edge to H from $L \in B$ to $K \in A$ if F is an endblock of L and $p(y_F) \in K$. Note again that every limb in H has outdegree at least 1 as it contains an endblock.

We now prove the analogue of Lemma 4.1. The original proof is complicated by the fact that endblock paths can join into the interior of endblocks, as discussed above.

Lemma 7.4. *Let G, a, b, G_a and G_b be as in Lemma 7.1. Suppose that G does not contain an $a-b$ path of length at least 2^d-1. Then H cannot contain an undirected path of length three.*

Proof. The proof of Lemma 4.1 applies unchanged if we can guarantee that for any endblock E of G_a or G_b, $p(x_E) \notin \text{int}(F)$ for any endblock F of G_b or G_a. Using Proposition 7.3 above we can therefore focus on the case where $\text{Body}(a) = \{a\}$ and $\text{Body}(b) = \{b\}$ i.e. a is a cutvertex of G_a and b is a cutvertex of G_b.

Suppose for contradiction that H contains a path of length at least 3. If one of the interior vertices on this path has two out-neighbours in Q the same argument as in the original proof will create a path between two exit vertices in this limb which extends two endblock paths of length $2^{d-1}-1$. This gives a path of length at least 2^d-2. Extending this path through Q as in Theorem 4.1 gives us an $a-b$ path of length at least 2^d-1. Similarly, if both endvertices on this path have out-neighbours in Q (that is $\overrightarrow{V_0V_1}$ and $\overleftarrow{V_2V_3}$ are edges of Q) we can find paths of length at least $2^{d-1}-1$ in both V_0 and V_3, which again can be joined through Q to give an $a-b$ path of length at least 2^d-1. This just leaves the case of a directed path

$$\overrightarrow{V_0V_1}, \overrightarrow{V_1V_2} \text{ and } \overrightarrow{V_2V_3}. \tag{1}$$

While here we obtain a path of length at least $2^{d-1}-1$ from the edge $\overrightarrow{V_0V_1}$ as before, in V_1, V_2, V_3 we might not be able to guarantee a full endblock path. Indeed, there is now the possibility that the edges $\overrightarrow{V_{i-1}V_i}$ and $\overrightarrow{V_iV_{i+1}}$ correspond to an edge entering an endblock E by a vertex x in its interior and the other edge leaving E by a vertex y in its interior. This does not allow us to apply Theorem 2.1 to E as $\mathrm{cutv}(E) \in E - \{x, y\}$ may have degree lower than $d-1$ in E. If, however, this does not happen at one of V_1 or V_2 the same proof applies.

We may also assume that the exit vertex x of V_2 guaranteed by $\overrightarrow{V_2V_3}$ has $p(x) \notin \mathrm{int}(F)$ for any endblock F in the interior of an endblock of V_3. Otherwise we can find an endblock path in V_0 of length at least $2^{d-1}-1$ and one in V_3 of length at least $2^{d-1}-2$. Since joining both of these through V_1 and V_2 joins at least four more vertices onto these paths, we can extend them to form an $a-b$ path of length at least 2^d+1, more than required.

But now take any outneighour of V_3 in H. Combined with our path Q above it is easily seen we can obtain a path Q' of length three which is either (i) not of the form (1), or (ii) contains V_3 as an interior vertex and allows for a full endblock path to be built through it. In both cases we are done. □

The following gives an analogue of Lemma 4.2. The proof from Lemma 4.2 applies unchanged, on noticing that by Proposition 7.3, exit vertices of endblocks of G_a and G_b again cannot have partners in the interior endblocks of G_b and G_a.

Lemma 7.5. *Let G, a, b, G_a and G_b be as in the statement of Lemma 7.1. Suppose G does not contain an $a-b$ path of length at least 2^d-1. Furthermore, suppose that Body$(a) \neq \{a\}$. Then no connected component of H contains two vertices of A.*

Combining Lemma 7.4 with Lemma 7.5 as in Corollary 4.3, we obtain the following:

Corollary 7.6. *Let G, a, b, G_a and G_b be as above. Suppose that G does not contain an $a-b$ path of length 2^d-1. Then the interaction digraph H of G has at least two connected components, one of which C consists entirely of limbs.*

Lemma 7.7. *Let G, a, b, G_a and G_b be as in the statement of Lemma 7.1. Suppose G does not contain an $a-b$ path of length at least 2^d-1. Then taking C as in Corollary 7.6, G_C has a 2-connected subgraph J containing two vertices $a' \in G_a$ and $b' \in G_b$ with the following properties:*

(i) *every vertex* $v \in J - \{a', b'\}$ *has degree at least* $d - 1$ *in* J *and all the neighbours of* v *in* G_C *are contained in* J

(ii) *for any vertex* $v \in (J - \{a', b'\}) \cap G_a$*,* J *contains an* $a' - v$ *path not containing* b *of length at least* $2^{d-1} - 2$*. Furthermore, if* v *has a neigbhour outside of* J*,* J *contains a* $b' - v$ *path not containing* a *of length at least* 2^{d-1}

(iii) *for any vertex* $v \in (J - \{a', b'\}) \cap G_b$*,* J *contains a* $b' - v$ *path not containing* a *of length at least* $2^{d-1} - 2$*. Furthermore, if* v *has a neighbour outside of* J*,* J *contains an* $a' - v$ *path not containing* b *of length at least* 2^{d-1}*.*

Proof. The proof of this lemma is almost identical to that of Lemma 5.2 with Lemma 7.4 and 7.5 taking the place of Lemma 4.1 and 4.2.

The only change to the argument of the proof is that in order to guarantee that we have $|S_i| \geq 2$ for some $i \in [t]$, we cannot now guarantee that $p(x_E)$ does not lie in the interior of any endblock of G_b. Instead, if $p(x_E) \in F$ for some endblock F of G_b, by Proposition 7.3 $\text{int}(F)$ contains an exit vertex y_F with $p(y_F) \neq x_E$. This again shows that $|S_i| \geq 2$ for some $i \in [t]$ and therefore that $t' \geq 1$.

The bounds in (ii) and (iii) follow from our new bounds on the length of endblock paths. Indeed, suppose $v \in (J - \{a', b'\}) \cap G_b$ say. Both the $a' - v$ and $b' - v$ paths in Lemma 5.2(ii) contain entire endblock paths and therefore have length at least $2^{d-1} - 2$. This gives the bound on the $b' - v$ path claimed. To obtain the $a' - v$ path, first note that as v has a neigbhour w outside J, by (i) it must be outside G_C. But then w must be v's partner, i.e. $w = p(v)$. Now in the last paragraph of the proof of Lemma 5.2, the endblock E is on the opposite side of J from v. But the path constructed in Lemma 5.2 combines a path from v to E with an endblock path in E. The first of these has to have length at least 2 as $p(v) \notin E$ and the second has length at least $2^{d-1} - 2$. Combining these we obtain an $a' - v$ path of length at least 2^{d-1}, as claimed. □

We will now give the proof of Lemma 7.1.

Proof of Lemma 7.1. Suppose for contradiction that G does not contain an $a - b$ path of length at least $2^d - 1$. Then by Corollary 7.6 the interaction digraph H of G contains a component C consisting entirely of limbs.

Now as G does not contain an $a - b$ path of length at least $2^d - 1$, we can apply Lemma 7.7 to find a 2-connected subgraph J of G_C and vertices a' and b' which satisfy Lemma 7.7 (i), (ii) and (iii). Again $|J| < |G|$.

Suppose first that $v \in J - \{a', b'\}$ has a neighbour w outside of J. Without loss of generality take $v \in G_b$. Then $w \notin G_C$ by Theorem 7.7(i) and so $w \in K$ for some limb K of a or $w \in \text{Core}(a)$. We will first deal with the case where $w \in K$.

If $w \notin \text{int}(E)$ for some endblock E of K take the $w - x_E$ path P_2 in K of length at least $2^{d-1} - 1$ given by Lemma 7.2(i). Combining this with the path P_1 given from Lemma 7.7(ii) in J from a' to v of length $2^{d-1} - 2$ and the edge $x_E p(x_E)$, we have a path $P_1 vwP_2 x_E p(x_E)$ from a' to $p(x_E)$ of length at least $(2^{d-1} - 2) + 1 + (2^{d-1} - 1) + 1 = 2^d - 1$. But this path extends to a path from a to b, a contradiction.

If $w \in \text{int}(E)$ for some endblock E of K, we would like to combine the $\text{Joint}(K) - w$ path guaranteed by induction on E with the $v - b'$ path in J as given by Lemma 7.7(ii) using the edge wv. This path extends to an $a-b$ path but may only have length $(2^{d-1}-2)+1+(2^{d-1}-2)=2^d-3$, too little for us.

Instead, look at an outneighbour of K in H. Let $K \in C'$ for some connected component $C' \neq C$ of H. If this is outneighbour is a limb, then C' must consist entirely of limbs, by Lemma 7.5. Therefore since all limbs have at least one outneighbour in H, C' contains a path of the form $\overleftarrow{KW}$ or of the form KVW where $\overleftarrow{VW}$ is an edge of H. This allows us to build a path P from w to $\mathrm{Joint}(W)$ of length at least $2^{d-1}+1$ in C'. Combining P with an appropriate path from Lemma 7.7(ii) via the edge vw we obtain a path that extends to an $a-b$ path of length at least $(2^{d-1}-2)+1+(2^{d-1}+1)=2^d$, as required – take this path to be the $a'-v$ path if $W \in B$ or the $b'-v$ path if $W \in A$.

If the outneighbour of K in H is instead $\mathrm{Core}(b)$, again using Lemma 7.7(ii) we can find a $b'-v$ path in J of length at least $2^{d-1}-2$ which extends through C' to give a $y-b'$ path P_2 of length at least 2^{d-1}, where $y \in \mathrm{Body}(b)$. Now H must have a third connected component C'' containing a limb of b since b has at least two limbs and only one element of B can lie in a component by Lemma 7.5. This component gives an $a-z$ path P_1 of length at least $2^{d-1}-2$ where again $z \in \mathrm{Body}(b)$ and P_1 and P_2 are disjoint. As in the proof of Lemma 4.1 we can join P_1 and P_2 together in $\mathrm{Core}(a)$ with a small use of 2-connectivity to give an $a-b$ path of length at least 2^d-1 as required. This completes the case when $w \in K$. The case where $w \in \mathrm{Core}(a)$ follows a similar argument, modifying the corresponding part of the proof of Lemma 3.4.

So we can assume that no vertex $v \in J-\{a',b'\}$ has an edge outside J. Then $d_J(v) \geq d$ for all $v \in J-\{a',b'\}$ and as $|J|<|G|$ we can apply Theorem 2.1 to J to find a path of length at least 2^d-2 from a' to b'. Moreover, unless J is isomorphic to Q_d with $a'=a$ and $b'=b$ where a and b are at even Hamming distance J contains a path of length at least 2^d-1 between a and b, so we may assume this is the case. Since G is not isomorphic to Q_d, the graph $G'=G[V(G)-J\cup\{a,b\}]$ is non-empty and all v in $G'-\{a,b\}$ have degree at least d in G'.

If a and b both have more than two limbs in G', G' is 2-connected. Then as $|G'|<|G|$ we can apply Theorem 2.1 to G'. This gives an $a-b$ path in G' of length at least 2^d-1 unless G' is isomorphic to Q_d. Now if G' was isomorphic to Q_d then J and G' would both contain the subcube containing a and b, which has at least four points since a and b are at even Hamming distance. But from construction G' and J only share a and b, so G' is not isomorphic to Q_d and therefore contains an $a-b$ path of length at least 2^d-1, a contradiction.

So one of a and b has exactly one limb. Let this be a say. Then $G'=G_{C'}$ for some component C' of H as all limbs of b must have an out-neighbour in H. Again we can apply Theorem 7.7 to G' to obtain a 2-connected subgraph $\widetilde{J}$ and vertices $\widetilde{a} \in G'_a$ and $\widetilde{b} \in G'_b$. As in Lemma 7.7(i) for any $v \in \widetilde{J}-\{\widetilde{a},\widetilde{b}\}$, $\widetilde{J}$ contains all neighbours of v in $G_C=G'$. As such v can have no neighbours in G other than those in G' we have $d_{\widetilde{J}}(v)=d_G(v)\geq d$. Theorem 2.1 holds for $\widetilde{J}$ taking $\widetilde{a}$ and $\widetilde{b}$ in place of a and b. This shows that $\widetilde{J}$ contains a $\widetilde{a}-\widetilde{b}$ path of length at least 2^d-2 which extends to an $a-b$ path in G' of length at least 2^d-2. As above, since J and $\widetilde{J}$ cannot both be isomorphic to Q_d if a and b are at even Hamming distance, one again must contain an $a-b$ path of length at least 2^d-1. This contradicts our assumption and proves the Lemma. □

Lastly, we show that the degree condition can again be removed.

Lemma 7.8. *Let G be a 2-connected subgraph of Q_n with $a, b \in G$ such that $d_G(a) = 2$ and $d_G(v) \geq d$ for all $v \in V(G) - \{a, b\}$, where $d \geq 3$. Suppose that Theorem 2.1 holds for smaller degrees and for all graphs G' with $|G'| < |G|$. Furthermore, suppose that G does not contain an $a - b$ path of length at least $2^d - 1$. Then the following hold:*

(i) *$G - a$ is a 2-connected graph.*

(ii) *There is a splitting direction i for G_a and G_b so that $d_{G_a}(a) \geq 2$ or $d_{G_b}(b) \geq 2$.*

As the proof is identical to that of Lemma 6.1 we will not repeat it. We can now finally give the proof of Theorem 2.1.

Proof of Theorem 2.1. The proof is the same as that of Theorem 2.3 up until the construction of the interaction digraph H. Let a, a', b, v and E_v be as in the proof of Theorem 2.3.

We again take H to be the bipartite multidigraph $H = (A', B, \overrightarrow{E})$ where $A' = \{K_1, \ldots, K_r\}$ and $B = \{L_1, \ldots, L_s\}$, the set of limbs of a' in G_a and b in G_b respectively. We also adjoin Core(b) to B if it is non-empty. Now from Lemma 7.2(i), for each endblock E in a limb K of a, $K \neq \{a, a'\}$, E contains an exit vertex x_E such that $p(x_E) \neq b$ and E contains a path of length at least $2^{d-1} - 1$ from cutv(E) to x_E. Pick one such exit vertex x_E for each endblock E other than $\{a, a'\}$ of G_a and such an exit vertex y_F for each endblock F of G_b. Also let $x_{E_v} = v$. For each endblock E of G_a, $E \neq \{a, a'\}$, adjoin a directed edge to H from $K \in A'$ to $L \in B$ if E is an endblock of K and $p(x_E) \in L$. Similarly, for each endblock F of G_b, adjoin a directed edge to H from $L \in B$ to $K \in A'$ if F is an endblock of L and $p(y_F) \in K$. Again every limb other than $\{a, a'\}$ has an outneighbour in H.

We now claim that we have a stronger analogue of Propostion 7.3 in this case, namely:

Claim 3: $p(x_E) \notin \text{int}(F)$ for all endblocks E and F, $E \neq E_v$.

Indeed, if this were the case, then E would contain a cutv$(E) - x_E$ path P_1 of length at least $2^{d-1} - 1$ and F would contain a $p(x_E) -$ cutv(F) path P_2 of length at least $2^{d-1} - 2$. Combining P_1 and P_2 with the $x_E p(x_E)$ and extending to a' and b we have an $a' - b$ path of length at least $2^d - 2$. Appending the edge aa' to this, G contains an $a - b$ path of length at least $2^d - 1$, a contradiction.

This claim allows us to establish Lemma 7.4 and 7.5 with the same proofs as in Lemma 4.1 and 4.2. Using this we again find a component C of H consisting entirely of limbs and not containing $\{a, a'\}$.

Now apply Lemma 7.7 again to G_C to find J with properties (i)-(iii). As a' is the neigbhour of a, we will write a_J and b_J for the vertices of J guaranteed by Lemma 7.7. In this J we can actually always guarantee that both of the paths in (ii) and (iii) have length at least 2^{d-1}. To see this, note that we can replace Claim 1 in the proof of 5.2 with the following:

Claim 4: $J - b$ has a 2-connected subgraph J' containing all of $J \cap G_a$ and an endblock F of G_b.

This claim is immediate from the proof of Lemma 5.2 on noticing that, by Claim 3, $p(S_i)$ cannot lie entirely in the interior of an endblock of G_b and so $\mathrm{span}_{G_b}(S_i) - \{b\}$ must contain an endblock of G_b.

Now use J' as in the final paragraph of the proof of Lemma 5.2 and choose one of E or F in place of E so that v is on the opposite side of J' to the chosen endblock; choose E if $v \in G_b$ and F if $v \in G_a$. The path then constructed contains an endblock path of length at least $2^{d-1} - 2$ and a path of length at least 2 from v to the chosen endblock. This gives the claimed path of length at least 2^{d-1}.

We now use this stronger fact to complete the proof. If there are no edges from $v \in J - \{a_J, b_J\}$ to a vertex in $G - J$ then, by induction on Theorem 2.1, J contains an $a_J - b_J$ path of length at least $2^d - 2$. Extending this to an $a' - b$ path and appending the edge aa' we obtain an $a - b$ path of length at least $2^d - 1$, a contradiction.

So we can assume that some $v \in J - \{a_J, b_J\}$ has a neighbour w outside J. The proof can now be finished in exactly the same way as the proof of Lemma 3.4. As here we always adjoin one of the paths in J of length at least 2^{d-1} with another endblock path of length at least $2^{d-1} - 2$ via the edge vw, the $a - b$ path we create always has length at least $2^d - 1$. This contradiction proves the Theorem. □

8 Generalizations

The reader might have noticed that we have used very little about Q_n in the proof of Theorem 2.1. The n-dimensional grid $\mathbb{Z}^n$ is the graph whose vertex set consists of n-tuples with entries in $\mathbb{Z}$ and in which two vertices x and y are adjacent if $|x_i - y_i| = 1$ for some $i \in [n]$ and $x_j = y_j$ for all $j \neq i$. The next theorem extends Theorem 2.1 (and therefore Theorems 1.1 and 1.3) to subgraphs of $\mathbb{Z}^n$.

Theorem 8.1. *Let G be a 2-connected subgraph of $\mathbb{Z}^n$ and $a, b \in V(G)$. Suppose that $d(z) \geq d$ for all $z \in V(G) - \{a, b\}$. Then a and b are joined by a path of length at least $2^d - 2$. Furthermore unless G is isomorphic to Q_d with a and b at even Hamming distance from each other, G contains an $a - b$ path of length $2^d - 1$.*

Proof. The crucial property of $\mathbb{Z}^n$ here is that we can always find a splitting of G into two connected pieces, G_a and G_b with $a \in G_a$ and $b \in G_b$ such that $d_{G_a}(a) \geq 1$ and $d_{G_b}(b) \geq 1$ and all $v \in G$ lose at most one neighbour in their piece. Indeed, taking some coordinate j on which a and b differ, say with $a_j > b_j$, let G_1 be the induced subgraph of G consisting all vertices v with $v_j \geq a_j$ and G_2 be the induced subgraph of G consisting of all w for which $w_j < a_j$. Again with the same modification to these graphs as in Lemma 2.2 we obtain connected graphs G_a and G_b with the required degree conditions. From here on the proof is identical to that of Theorem 2.1. □

Moreover, the same proof also extends to subgraphs of the discrete torus C_k^n provided $k \geq 4$. Now we cannot expect a bound of the form $C2^d$ as above for subgraphs of the discrete torus C_3^d as this graph has minimum degree $2d$ but only 3^d points. This shows that given a subgraph G of C_3^n of minimal degree at least d we cannot in general guarantee a path of length more than $3^{\frac{d}{2}} - 1$ in G.

Why does our approach not work in this case? The main reason is that we cannot guarantee a partition into two subgraphs such that all vertices lose at most one neighbour in their piece. Can we still guarantee an exponentially long path in this case?

The following general result shows that we can.

Theorem 8.2. *Let $k \in \mathbb{N}$ and G be a 2-connected graph with $a, b \in V(G)$. Suppose $d(v) \geq d$ for all $v \in V(G) - \{a, b\}$. Furthermore, suppose that G has the following property:*

> *Given any two vertices $x, y \in G$, there is a partition of $V(G)$ into two sets X and Y with $x \in X$ and $y \in Y$ such that $d_{G[X]}(v) \geq d(v) - k$ for all $v \in X$ and $d_{G[Y]}(v) \geq d(v) - k$ for all $v \in Y$.*

Then G contains an $a - b$ path of length at least $2^{\frac{d}{k+2}}$.

Note that if the property above holds for G, it also holds for all subgraphs of G. Also note that Theorem 1.4 immediately follows from Theorem 8.2. As an immediate corollary of Theorem 8.2 we have the following:

Corollary 8.3. *Every subgraph of C_3^n of minimum degree at least d contains a path of length at least $2^{\frac{d}{4}}$.*

It would be interesting to decided what the correct lower bounds for the length of the longest path in subgraphs of C_3^n with minimum degree at least d.

Conjecture 8.4. *Given a subgraph G of C_3^n with minimum degree at least d, G must contain a path of length at least $3^{\frac{d}{2}} - 1$.*

Another consequence of Theorem 8.2 is the following result for product graphs.

Theorem 8.5. *Let $G_1, \ldots, G_l$ be graphs with maximum degree at most k. Then given any subgraph G of the Cartesian product graph $\prod_{i=1}^{l} G_i$ of minimum degree at least d, G contains a path of length at least $2^{\frac{d}{k+2}}$.*

The proof of Theorem 8.2 is similar to that of Theorem 2.1 but shorter.

Proof. The proof is again by induction on d. It suffices to prove the result for $d \geq k+4$ as otherwise it follows from 2-connectivity. As in the proof of Theorem 2.1 we wish to split G into two subgraphs G_a and G_b with $a \in G_a$ and $b \in G_b$, which is the motivation for the above splitting property. However, simply taking a and b in place of x and y might not be useful as both a and b can have degree as low as two in G in which case in the partition guaranteed above a may end up with all its neighbours in Y. Instead we pick a neighbour $a' \neq b$ of a and a neighbour $b' \neq a$ of b. The fact that G is 2-connected ensures it is possible to pick $a' \neq b'$. Now take the partition guaranteed from our splitting property above with $x = a'$ and $y = b'$. Moving a to X and b to Y as needed we have that $d_{G[X]}(v) \geq d(v) - k - 1$ for all $v \in X - a$ and similarly for $v \in Y - b$. Both a and b now have at least 1 neighbour in $G[X]$ and $G[Y]$ respectively. Finally, denoting the connected component of $G[X]$ containing a by C_a, let G_b be the connected

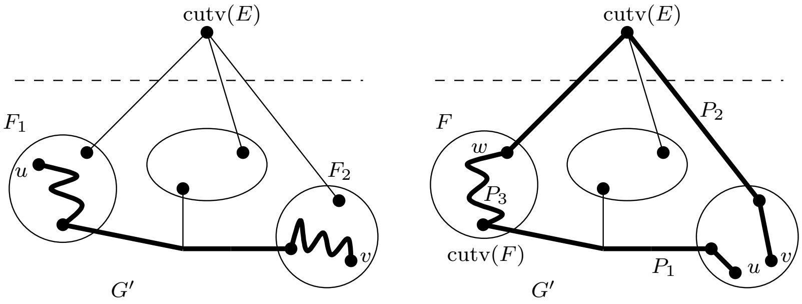

Figure 5: Cases where G' is not 2-connected in Lemma 8.6

component of $G - C_a$ containing b and $G_a = G[V(G) - V(G_b)]$. Note that G_a and G_b are connected with $a \in G_a$, $b \in G_b$. Moreover, $d_{G_a}(v) \geq d(v) - k - 1$ for all $v \in G_a - a$ and $d_{G_b}(v) \geq d(v) - k - 1$ for $v \in G_b - b$.

We will again analyse the block-cutvertex decompositions of G_a and G_b. The following lemma will be very useful below.

Lemma 8.6. *Let E be an endblock of G_a or G_b with $a, b \notin int(E)$. Then given any two vertices $u, v \in int(E)$, $G[E]$ contains a path of length at least $2^{\frac{d-k-2}{k+2}}$ from u to v.*

Proof. Look at the block-cutvertex decomposition of $G' = G[E] - \mathrm{cutv}(E)$. Since E is 2-connected (as $d \geq k+4$), G' is connected and $\mathrm{cutv}(E)$ must have a neighbour in the interior of every endblock of G'. Note that every vertex $v \in G'$ has $d_{G'}(v) \geq d_G(v) - k - 2$. In particular, since $d \geq k+4$ each endblock F of G' is 2-connected and has at least three vertices so that we can by induction apply Theorem 8.2 to it. If G' is 2-connected then by induction on Theorem 8.2 G' contains the desired path from u to v. Thus we may assume that G' is not 2-connected. If $u \in \mathrm{int}(F_1)$ and $v \in \mathrm{int}(F_2)$ where F_1 and F_2 are two distinct endblocks of G' then by induction on Theorem 8.2, $G[F_1]$ and $G[F_2]$ contain $u - \mathrm{cutv}(F_1)$ and $\mathrm{cutv}(F_2) - v$ paths respectively, each of length at least $2^{\frac{d-k-2}{k+2}}$. Joining $\mathrm{cutv}(F_1)$ to $\mathrm{cutv}(F_2)$ by a third path in G' and combining all three of these paths, we get a $u - v$ path of length at least $2^{\frac{d}{k+2}}$, as required. Therefore since G' contains at least two endblocks, we can assume that one of these, say F, does not contain u or v in its interior. Contracting $\mathrm{int}(F)$ down to a single vertex in $G[E]$, the resulting graph is still 2-connected. Therefore, as in the proof of Lemma 5.2, $G[E]$ contains two vertex disjoint paths P_1 and P_2 from the set $\{u, v\}$ to $\{\mathrm{cutv}(F), w\}$ for some $w \in \mathrm{int}(F)$, with $(P_1 \cup P_2) \cap (F - \{\mathrm{cutv}(F), w\}) = \emptyset$. Now using induction on Theorem 8.2 in F, it contains a path P_3 of length $2^{\frac{d-k-2}{k+2}}$ from $\mathrm{cutv}(F)$ to w. Piecing P_1, P_2 and P_3 together we obtain our desired path. $\square$

Again we have:

Proposition 8.7. *Let E be an endblock of G_a not containing a and F an endblock of G_b not containing b. Then G does not contain an edge from $int(E)$ to $int(F)$*

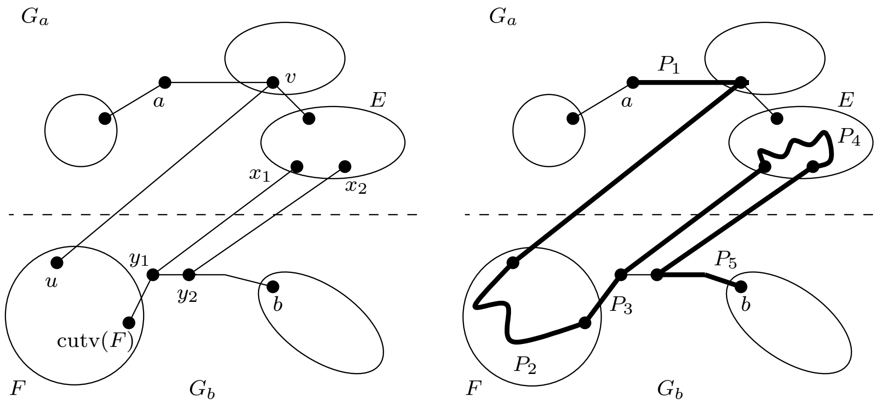

Figure 6: Path created in Theorem 8.2

Proof. Exactly as in Proposition 3.6. □

Lemma 8.8. *We have the following:*

(i) *Given any endblock E of G_a not containing a, there are two disjoint edges from int(E) to G_b in G.*

(ii) *G_a contains an endblock not containing a.*

Proof. (i) E must have an exit vertex x_1, with neighbour $y \in G_b$, as G is 2-connected. If it had only one, $G' = G[E]$ is 2-connected and every $v \in G' - \{x_1, \mathrm{cutv}(E)\}$ has degree at least d. Therefore, by induction on Theorem 8.2, G' contains a path of length at least $2^{\frac{d}{k+2}}$ from x_1 to $\mathrm{cutv}(E)$. Extending this path from $\mathrm{cutv}(E)$ to a in G_a and from y to b in G_b we obtain an $a-b$ path of desired length. Therefore we may assume E contains a second exit vertex x_2. Now if the vertices in $\mathrm{int}(E)$ were only adjacent to y in G_b, x_1y and x_2y must be edges of G. Then $G'' = G[E \cup \{y\}]$ is 2-connected and $d_{G''}(v) \geq d$ for every $v \in G'' - \{y, \mathrm{cutv}(E)\}$. By Theorem 8.2 G'' contains a $\mathrm{cutv}(E) - y$ path of length at least $2^{\frac{d}{k+2}}$. Again, extending this to a path from a to b, we have an $a-b$ path of length at least $2^{\frac{d}{k+2}}$. Therefore we may assume the two edges exist or we are done.

(ii) The proof is almost identical to the proof of Lemma 3.5(ii). □

Take an endblock E of G_a not containing a, as guaranteed by Lemma 8.8(ii). We can choose E such that a and all $v \in G_a - E$ not contained in the interior of an endblock of G_a lie in the same connected component of $G_a - E$ (e.g. pick a block B in G_a containing a and choose E to be a block at maximum distance from B in $\mathcal{B}(G_a)$). Let x_1y_1 and x_2y_2 be the disjoint edges of G with $x_1, x_2 \in \mathrm{int}(E)$ and $y_1, y_2 \in G_b$ guaranteed by Lemma 8.8(i). By Proposition 8.7, $y_1, y_2 \notin \mathrm{int}(F)$ for all endblocks F of G_b not containing b in its interior.

Now looking at the block-cutvertex decomposition of G_b we can choose two vertex disjoint paths in G_b from $\{y_1, y_2\}$ to $\{b, \mathrm{cutv}(F)\}$ where F is some endblock of G_b not containing b. Lets say that these paths are P_3 from $\mathrm{cutv}(F)$ to y_1 and P_5 from y_2 to b. Applying Lemma 8.8(i) to F we see that there exists

$u \in \mathrm{int}(F)$ adjacent to some $v \in G_a$, $v \neq \mathrm{cutv}(E)$. Furthermore, by Proposition 8.7 $v \notin \mathrm{int}(E')$ for any endblock E' of G_a. From our choice of E there exists an $a-v$ path P_1 in $G_a - E$. Finally by induction on Theorem 8.2, F contains a $u - \mathrm{cutv}(F)$ path P_2 of length at least $2^{\frac{d-k-1}{k+2}}$ and by Lemma 8.6 E contains an x_1x_2 path P_4 of length at least $2^{\frac{d-k-2}{k+2}}$. Combining these five paths we obtain an $a-b$ path $P = P_1vuP_2P_3y_1x_1P_4x_2y_2P_5$ of length at least $2^{\frac{d-k-1}{k+2}} + 2^{\frac{d-k-2}{k+2}} > 2^{\frac{d}{k+2}}$ as required. □

The cycle analogues of the above theorems can be obtained in a similar fashion to the proof of Theorem 1.3 from Theorem 2.1.

As mentioned in the Introduction, we do not know the correct bound for the length of the longest path in a subgraph of Q_n when the minimum degree condition in Theorem 1.1 is replaced by an average degree condition. Is the following possible?

Conjecture 8.9. *Every subgraph of Q_n with average degree at least d contains a path of length at least $2^d - 1$.*

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